

FUNDAMENTAL SOLUTION OF THE PROBLEM OF LINEAR PROGRAMMING AND
METHOD OF ITS DETERMINATION

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FUNDAMENTAL SOLUTION OF THE PROBLEM OF LINEAR PROGRAMMING AND METHOD OF ITS DETERMINATION

S. V. Petrunin

ABSTRACT. The idea of a fundamental solution to a problem in linear programming is introduced. We propose a method of determining the fundamental solution and of applying this method to the solution of a problem in linear programming. Numerical examples are cited.

1. The Notion of a "Fundamental" Solution

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We consider a problem in linear programming (LP) in the following form. It is required that we minimize the linear form in n variables x_i :

$$Z = \sum_{i=1}^n c_i x_i, \quad i = \overline{1, n} \quad (1)$$

under the linear constraints:

$$\sum_{i=1}^n a_{ji} x_i = b_j, \quad j = \overline{1, m}, \quad m < n, \quad (2)$$

$$x_i \geq 0. \quad (3)$$

* Number in the margin indicate pagination in the original foreign text.

All methods existing at the present time seek a final collection of variables $\{x_i\}$ which satisfy (2) and (3) and minimize Z .

It is not practical in any of these to make use of the fact that some of the x_i can be equal to zero for every b_j .

We will call a collection of variables x_i^* a "fundamental" solution if every one of these variables, even if only for one combination of the $\{b_j\}$ will be different from zero.

Before proceeding to the description of the method for finding a fundamental solution, we observe that with every problem in linear programming, we can compare another LP problem, called the dual [1]. /4

In particular, for the problem (1) - (3), the dual will be the problem in which it is required to find the maximum of the linear form in the variables y_1, \dots, y_m :

if
$$V = \sum_{j=1}^m b_j y_j \quad j = \overline{1, m}, \quad (4)$$

$$\sum_{j=1}^m a_{ji} y_j \leq c_i \quad i = \overline{1, n}. \quad (5)$$

Condition (5) with the addition of new variables $\omega_i \geq 0$ can be written in the form:

$$\sum_{j=1}^m a_{ji} y_j + \omega_i = c_i, \quad \omega_i \geq 0, \quad i = \overline{1, n}. \quad (6)$$

The optimal values of the variables w_i and x_i , in accordance with the theorem on weak complementary slackness [2], possesses the property that:

$$x_i \cdot \omega_i = 0, \quad i = \overline{1, n}.$$

Moreover, if for the direct and dual problems there exist feasible solutions, then according to the theorem on strong complementary slackness [2], we can find at least one pair of variables x_2 and w_2 such that, if $w_2 = 0$, then $x_2 > 0$.

These properties are used for finding the fundamental solution.

2. The Method of Determining the Fundamental Solution

We turn directly to a description of the method. We write the constraints (2) in the form of a canonical system which, according to [1], can be described thus:

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"A canonical system with an ordered subset of variables, called basis variables, is a system where, for each j , the j^{th} basis variable has coefficient one in the j^{th} equation and zero coefficients in the other equations."

In the canonical form, an LP problem can be formulated thus:

To find the minimum of

$$Z = \sum_{i=1}^k c_i x_i \quad (7)$$

under the constraints

$$\begin{aligned} x_j &= b_j + \sum_{i=1}^k a_{ji} x_i, \quad j = k+1, n. \\ x_i, x_j &\geq 0. \end{aligned} \quad (8)$$

We represent all the independent variables x_i in the form:

$$x_i = L_i + n_i y, \quad (9)$$

where L_i and y — new variables,
 n_i — some parameters.

The basis variables x_j and the objective function Z can be written:

$$x_j = d_j + y \sum_{i=1}^K a_{ji} n_i, \quad (10)$$

$$Z = \sum_{i=1}^K c_i d_i + y \sum_{i=1}^K c_i n_i, \quad (11)$$

where

$$d_j = b_j + \sum_{i=1}^K a_{ji} d_i.$$

Combining equations (9) and (10):

$$\begin{aligned} -n_i y + x_i &= d_i, & i = \overline{1, K}, \\ -y \sum_{i=1}^K a_{ji} n_i + x_j &= d_j, & j = \overline{K+1, n}, \end{aligned}$$

or because $x_i, x_j \geq 0$:

$$-n_i y \leq d_i, \quad i = \overline{1, K}. \quad (12)$$

$$-y \sum_{i=1}^K a_{ji} n_i \leq d_j, \quad j = \overline{K+1, n}. \quad (13)$$

We assume that in (12) and (13), d_i and d_j have constant values corresponding to the minimum of Z . Then, comparing (12) and (13) with (5), we see that, due to the arbitrariness of y , expressions (12), (13), and (11) represent the dual problem for the following primal problem, in which is sought:

$$\min \left\{ Z = \sum_{i=1}^K d_i x_i + \sum_{j=K+1}^n d_j x_j \right\}$$

(the objective function is written with precision up to a constant) in the presence of the one condition:

$$\sum_{i=1}^K n_i \gamma_i + \sum_{j=K+1}^K \left[\gamma_j \sum_{i=1}^K a_{ji} n_i \right] = \sum_{i=1}^K c_i n_i \quad (14)$$

$\gamma_i, \gamma_j \geq 0$

The values of the variables in the primal, as well as the dual, problem must be non-negative. Keeping this in mind, we construct equation (14) in such a manner that all the ratios

$$\frac{\sum_{i=1}^K c_i n_i}{n_i} \quad \text{and} \quad \frac{\sum_{i=1}^K c_i n_i}{\sum_{i=1}^K a_{ji} n_i}$$

except one, will be negative. Then the solution, if it exists, will be determined only by the term γ_m , for which the coefficient:

$$\frac{\sum_{i=1}^K c_i n_i}{n_m} \quad \text{or} \quad \frac{\sum_{i=1}^K c_i n_i}{\sum_{i=1}^K a_{mi} n_i}$$

will be positive. For such an equation, $\gamma_m > 0$, while the remaining $\gamma_i, \gamma_j = 0$. In the dual problem, however, $m_x = 0$, on the contrary, while the remaining $x_i, x_j \geq 0$. In this way, if we succeed

in finding a collection of coefficients n_i , such that all $\frac{\sum_{i=1}^K c_i n_i}{n_i}$ $\frac{\sum_{i=1}^K c_i n_i}{\sum_{i=1}^K a_{ji} n_i}$ (or $\frac{n_i}{\sum_{i=1}^K c_i n_i}$ and $\frac{\sum_{i=1}^K a_{ji} n_i}{\sum_{i=1}^K c_i n_i}$), except the

one for γ_m , will not be larger than zero; then we can set $x_m = 0$.

Here there can be two cases: either $m \leq K$ or $m > K$. In the first case, it suffices to set $n_m = 0$ and continue the analysis not changing the remaining coefficients. If $m > K$, then it follows that we can transform the initial system of equations by setting

$x_m = 0$ and making it one of the independent variables in the basis. By this procedure, the number of independent random variable decreases by one. For the new system we again construct the coefficients:

$$\sum_{i=1}^{k-1} c_i n_i^* / n_i^* \quad \text{and} \quad \sum_{i=1}^{k-1} c_i n_i^* / \sum_{i=1}^{k-1} a_{ji} n_i^*,$$

and we undertake the same work.

3. A Numerical Example

To find the minimum $Z = -x_1 + x_2 + x_3$ under the conditions:

$$x_4 = -x_1 + 2x_2 + x_3,$$

$$x_5 = 4 - x_1 - 5x_3.$$

we set:

$$x_1 = l_1 + n_1 y,$$

$$x_2 = l_2 + n_2 y,$$

$$x_3 = l_3 + n_3 y,$$

$$x_4 = l_4 + (-n_1 + 2n_2 + n_3) y,$$

$$x_5 = l_5 + (-n_1 - 5n_3) y.$$

$$Z = l_2 + (-n_1 + n_2 + n_3) y.$$

Then the auxiliary equation has the form:

$$n_1 y_1 + n_2 y_2 + n_3 y_3 + (-n_1 + 2n_2 + n_3) y_4 + (-n_1 - 5n_3) y_5 = -n_1 + n_2 + n_3.$$

It is necessary to require that one of the coefficients

$$\frac{n_1}{-n_1 + n_2 + n_3}, \frac{n_2}{-n_1 + n_2 + n_3}, \frac{n_3}{-n_1 + n_2 + n_3}, \frac{-n_1 + 2n_2 + n_3}{-n_1 + n_2 + n_3}, \frac{-n_1 - 5n_3}{-n_1 + n_2 + n_3},$$

be greater than zero and the rest — less than zero.

We assume that:

$$-n_1 + n_2 + n_3 > 0.$$

We first explain that the independent variables do not vanish. For this, we assume that:

$$\begin{aligned} -n_1 + 2n_2 + n_3 &< 0, \\ -n_1 - 5n_3 &< 0. \end{aligned}$$

We consider the system of inequalities:

$$\begin{aligned} -n_1 + n_2 + n_3 &> 0, & n_1 - n_2 - n_3 &< 0, \\ -n_1 + 2n_2 + n_3 &< 0, & \text{or} & -n_1 + 2n_2 + n_3 < 0, \\ -n_1 - 5n_3 &< 0, & -n_1 - 5n_3 &< 0. \end{aligned}$$

From the first two inequalities, it follows that $n_2 < 0$.

Adding the first and third inequalities, we have:

$$-n_2 - 6n_3 < 0.$$

Because $n_2 < 0$, this inequality will hold if $n_3 > 0$. In order to evaluate the sign of n_1 , we write:

$$\begin{aligned} n_1 &< n_2 + n_3, \\ n_1 &> 2n_2 + n_3, \\ n_1 &> -5n_3. \end{aligned}$$

From these inequalities, it follows that, if we assume $|n_2| > n_3$, /9
then we can consider that $n_1 < 0$. From this,

$$\begin{aligned} n_1 &< 0, \quad n_2 < 0, \quad n_3 > 0, \\ -n_1 + 2n_2 + n_3 &< 0, \quad -n_1 - 5n_3 &< 0. \end{aligned}$$

Therefore,

$$\gamma_3 > 0, \quad \gamma_1 = \gamma_2 = \gamma_4 = \gamma_5 = 0 \quad \text{and} \quad \gamma_3 \neq 0.$$

Returning again to the system of inequalities, setting $n_3 = 0$:

$$\begin{aligned} n_1 - n_2 &< 0, \\ -n_1 + 2n_2 &< 0, \\ -n_1 &< 0. \end{aligned}$$

These inequalities are inconsistent. We assume that $\gamma_5 > 0$.
Then,

$$\begin{aligned} n_1 - n_2 &< 0, \\ -n_1 + 2n_2 &< 0, \quad n_1 < 0, \\ -n_1 &> 0. \end{aligned}$$

for $n_2 < 0$, it follows from the first inequality that $|n_1| > |n_2|$,
and from the second that $2|n_2| > |n_1|$. Therefore, choosing n_1
such that:

$$2|n_2| > |n_1| > |n_2|, \quad n_1 < 0, \quad n_2 < 0,$$

we will have:

$$\gamma_5 > 0, \quad \gamma_1 = \gamma_2 = \gamma_4 = 0 \quad \text{and} \quad x_5 = 0.$$

We return to the original system of equations and assume

$$\begin{aligned} x_3 = 0, \quad x_5 = 0 \\ x_4 = -x_1 + 2x_2, \\ 0 = 4 - x_1. \end{aligned}$$

From this, $x_1 = 4$.

$$\begin{aligned} x_4 &= -4 + 2x_2, \\ x_2 &= -4 + x_2. \end{aligned}$$

We apply the proposed method to the system:

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$$\begin{aligned}x_2 &= \bar{x}_2 + n_2' y, \\x_4 &= \bar{x}_4 + 2n_2' y, \\z &= \bar{z} + n_2' y.\end{aligned}$$

The auxiliary equation has the form:

$$n_2' y_2 + 2n_2' y_4 = n_2',$$

which does not permit us to find the new variable not occurring in the fundamental solution. Consequently, x_1, x_2, x_4 comprise a fundamental solution for the matrix of conditions:

$$\begin{Bmatrix} -1, 2, 1, -1, 0 \\ -1, 0, -5, 0, -1 \end{Bmatrix}$$

and coefficients $C_i (-1, 1, 1, 0, 0)$ of the objective function.

Insofar as $x_1 = 4$ (identically) in the fundamental solution, the particular solution to the given problem, if one exists, will be either x_1, x_2 or x_4, x_5 .

4. Using the Method for the Complete Solution to an LP Problem. A Numerical Example

The example of the previous paragraph demonstrated sufficiently graphically the use of the method to obtain a fundamental solution. However, in many practical cases, the proposed method can be applied also to obtain the particular solution.

In fact, it was shown above that the fundamental solution to the problem depends only on the matrix of coefficients of the conditions a_{ji} and the coefficients c_i of the objective function, but it does not depend on the b_j . Consequently, applying the method only to the primal problem, we cannot find the particular solution, which depends also on the values of the b_j . We return to the dual

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problem and we find its fundamental solution. It naturally will depend on the matrix of coefficients of the conditions and (which for us is especially important) on the coefficients of the objective function of the dual problem. And these coefficients are functions of the values b_j of the primal problem [1]. Therefore, using the method on the primal and dual problems, we can obtain in a series of situations the particular solution to the problem. We emphasize that this method eliminates the possibility of cycling, which sometimes occurs in the iterative loop of the simplex method. As proof, we mention the solution to a problem having cycling in its solution by the simplex method (the so-called example of Beale [1]).

Example.

$$\begin{aligned} \frac{1}{4} x_1^* + 60 x_2^* - \frac{1}{25} x_3^* + 9 x_4^* + x_5^* &= 0, \\ \frac{1}{2} x_1^* + 90 x_2^* - \frac{1}{50} x_3^* + 3 x_4^* + x_6^* &= 0, \\ x_3^* + x_7^* &= 1, \\ z^* &= -\frac{3}{4} x_1^* + 150 x_2^* - \frac{1}{50} x_3^* + 6 x_4^*. \end{aligned}$$

We introduce the new variables:

$$\begin{aligned} x_1 &= \frac{25}{2} x_1^*, & x_5 &= 25 x_5^*, \\ x_2 &= 750 x_2^*, & x_6 &= 25 x_6^*, \\ x_3 &= x_3^*, & x_7 &= x_7^*, \\ x_4 &= 75 x_4^*, & z &= 25 z^*. \end{aligned}$$

Then,

$$\begin{aligned} x_5 &= -\frac{1}{2} x_1 + 2 x_2 + x_3 - 3 x_4, \\ x_6 &= -x_1 + 3 x_2 + \frac{1}{2} x_3 - x_4, \\ x_7 &= 1 - x_3, \\ z &= -\frac{3}{2} x_1 + 5 x_2 - \frac{1}{2} x_3 + 2 x_4. \end{aligned}$$

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We set:

$$\begin{aligned} x_1 &= \bar{x}_1 + n_1 y, & x_5 &= \bar{x}_5 + \left(-\frac{1}{2}n_1 + 2n_2 + n_3 - 3n_4\right)y, \\ x_2 &= \bar{x}_2 + n_2 y, & x_6 &= \bar{x}_6 + \left(-n_1 + 3n_2 + \frac{1}{2}n_3 - n_4\right)y, \\ x_3 &= \bar{x}_3 + n_3 y, & x_7 &= \bar{x}_7 - n_3 y, \\ x_4 &= \bar{x}_4 + n_4 y, & z &= \bar{z} + \left(-\frac{3}{2}n_1 + 5n_2 - \frac{1}{2}n_3 + 2n_4\right)y. \end{aligned}$$

We assume:

$$\begin{aligned} \frac{3}{2}n_1 - 5n_2 + \frac{1}{2}n_3 - 2n_4 &< 0, & [z] \\ \frac{1}{2}n_1 - 2n_2 - n_3 + 3n_4 &> 0, & [x_5] \\ n_1 - 3n_2 - \frac{1}{2}n_3 + n_4 &> 0, & [x_6] \\ n_3 &< 0. & [x_7] \end{aligned}$$

For $n_1 < 0$, $n_2 < 0$, $n_4 < 0$, these inequalities are valid.

Therefore, $x_7 > 0$, $x_7 = 0$. From the original system, $x_3 = 1$, and

$$\begin{aligned} x_5 &= 1 - \frac{1}{2}x_1 + 2x_2 - 3x_4, \\ x_6 &= \frac{1}{2} - x_1 + 3x_2 - x_4, \\ z &= -\frac{1}{2} - \frac{3}{2}x_1 + 5x_2 + 2x_4. \end{aligned}$$

We assume $n_3 = 0$, and continue to analyze the system of inequalities: /13

$$\begin{aligned} \frac{3}{2}n_1 - 5n_2 - 2n_4 &< 0, \\ -\frac{1}{2}n_1 + 2n_2 - 3n_4 &< 0, \\ -n_1 + 3n_2 - n_4 &< 0. \end{aligned}$$

The system is valid for $n_1 < 0, n_2 < 0, n_4 > 0$. Consequently, $y_4 > 0, x_4 = 0$. We assume

$$\begin{aligned} \frac{3}{2}n_1 - 5n_2 &< 0, \\ -\frac{1}{2}n_1 + 2n_2 &< 0, \\ -n_1 + 3n_2 &< 0. \end{aligned}$$

This system is valid neither for $n_1 > 0, n_2 < 0$, nor for $n_2 > 0, n_1 < 0$. Hence, one of the basis variables equals zero. We assume that $y_6 > 0$, i.e., $-n_1 + 3n_2 > 0$ for $n_1 < 0, n_2 < 0$.

$$\left. \begin{aligned} \frac{3}{2}n_1 - 5n_2 &< 0 \\ -\frac{1}{2}n_1 + 2n_2 &< 0 \\ -n_1 + 3n_2 &< 0 \end{aligned} \right\} \quad \begin{aligned} &\text{These inequalities are valid for} \\ &0,3n_1 < n_2 < 0,25n_1, \quad n_1 < 0, n_2 < 0. \end{aligned}$$

Consequently $y_6 > 0, x_6 = 0$, and the original system

$$\begin{aligned} x_5 &= 1 - \frac{1}{2}x_1 + 2x_2, \\ 0 &= \frac{1}{2} - x_1 + 3x_2. \end{aligned}$$

From this,

$$\begin{aligned} x_1 &= \frac{1}{2} + 3x_2, \\ x_5 &= \frac{3}{4} + \frac{1}{2}x_2, \quad Z = -\frac{5}{4} + \frac{1}{2}x_2. \end{aligned}$$

We denote again:

$$x_2 = l'_2 + n'_2 y.$$

Then,

$$\begin{aligned} x_1 &= l'_1 + 3n'_2 y, \\ x_5 &= l'_5 + \frac{1}{2}n'_2 y, \quad Z = l'_Z + \frac{1}{2}n'_2 y. \end{aligned}$$

and the auxiliary equation:

$$3n_2' \gamma_1 + n_2' \gamma_2 + \frac{1}{2} n_2' \gamma_5 = \frac{1}{2} n_2.$$

This equation does not permit us to define new variables equal to zero for arbitrary b_j . Consequently x_1, x_2, x_5 are the fundamental solution to the problem. To determine a particular solution, we construct the dual problem (the objective function is written with accuracy up to a constant):

$$\omega_2 + 3\omega_1 + \frac{1}{2}\omega_5 = \frac{1}{2},$$

$$v = -\frac{1}{2}\omega_1 - \frac{3}{4}\omega_5,$$

or

$$z^0 = \frac{1}{2}\omega_1 + \frac{3}{4}\omega_5.$$

We assume:

$$\omega_1 = z_1^0 + n_1^0 y, \quad \omega_2 = z_2^0 + (-3n_1^0 - \frac{1}{2}n_5^0)y,$$

$$\omega_5 = z_5^0 + n_5^0 y, \quad z^0 = z_z^0 + (\frac{1}{2}n_1^0 + \frac{3}{4}n_5^0)y.$$

The auxiliary equation:

$$n_1^0 \gamma_1^0 + (-3n_1^0 - \frac{1}{2}n_5^0) \gamma_2^0 + n_5^0 \gamma_5^0 = \frac{1}{2}n_1^0 + \frac{3}{4}n_5^0.$$

Let:

$$\left. \begin{aligned} \frac{1}{2}n_1^0 + \frac{3}{4}n_5^0 &> 0 \\ 3n_1^0 + \frac{1}{2}n_5^0 &> 0 \end{aligned} \right\}$$

These inequalities are valid if

$$n_5^0 > 0, \text{ but } n_1^0 < 0$$

$$\text{and } n_5^0 > -6n_1^0.$$

Because $n_5^0 > 0$, then $\gamma_5^0 > 0$ and $\omega_5 = 0$. Consequently,

$$3\omega_1 + \omega_2 = \frac{1}{2},$$

$$\bar{z}^0 = \frac{1}{2}\omega_1$$

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and the auxiliary equation:

$$w_1^0 y_1^0 - 3w_1^0 y_2^0 = \frac{1}{2} w_1^0.$$

Here, $y_1^0 > 0$ and $w_1 = 0$. Consequently, only $w_2 > 0$, and, returning to the primal problem, we have $x_2 = 0$. Then the particular solution will be:

$$x_1 = \frac{1}{2}, x_2 = 0, x_3 = 1, x_4 = 0, x_5 = \frac{3}{4}, x_6 = x_7 = 0.$$

or, in the original variables:

$$x_1^* = \frac{1}{25}, x_2^* = 0, x_3^* = 1, x_4^* = 0, x_5^* = \frac{3}{100}, x_6^* = x_7^* = 0.$$

$$\text{and } \lambda^* = -\frac{1}{20}.$$

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